ISRAEL JOURNAL OF MATHEMATICS, Vol. 48, No. 4, 1984

THE CLASSIFICATION OF INJECTIVE BANACH LATTICES

BY

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ABSTRACT

Let E be a 1-injective Banach lattice, X any Banach space and $T: E \to X$ a norm bounded linear operator. Then either T is an isomorphism on some copy of l_∞ in E or for all $\sigma > 0$ there is $\varphi \in E'_+$ such that $\| Tu \| \leq \varphi(|u|) + \sigma \| u \|$ for all $u \in E$. We deduce the theorem that: A norm order continuous injective Banach lattice is order isomorphic to an (AL)-space.

1. Introduction

The concept of a 1-injective Banach lattice first appears in Lotz [5] who proved that P_1-spaces and (AL)-spaces are 1-injective Banach lattices and that the class of 1-injective Banach lattices is closed with respect to direct sums in the sense of l_∞. In [1] Cartwright proves that every finite dimensional 1-injective Banach lattice is order isometric to a lattice of the form $(\Sigma_{j=1}^m \oplus l_1^{m_j})_\infty$: that is, a finite l_∞-direct sum of (finite dimensional) $L_1(\nu)$-spaces. And in [4] Lindenstrauss and Tzafriri have shown that there is a function $f(\lambda)$ such that every finite dimensional λ-injective Banach lattice $(\lambda > 1)$ is order isomorphic, with isomorphism constant $f(\lambda)$, to a lattice of the form $(\Sigma_{j=1}^m \oplus l_1^{m_j})_\infty$. It follows that any discrete injective Banach lattice is order isomorphic to a 1-injective Banach lattice. In [2] Haydon had obtained an operator-theoretic representation of general 1-injective Banach lattices.

These results prompt the following problem: What representations are available for general injective Banach lattices? More precisely: Is every injective Banach lattice order isomorphic to a 1-injective Banach lattice?

2. Preliminaries

2.1. Definitions. A real Banach lattice E is a λ-injective Banach lattice (in short: λ-IBL) if for every Banach lattice G, every closed sublattice F of G

Received October 1, 1982 and in revised form July 10, 1983

and every positive linear operator $u : F \to E$ there is a positive linear extension $v : G \to E$ with $\|v\| \leqq \lambda \|u\|$. An injective Banach lattice is a lattice that is a λ-IBL for some $\lambda > 1$.

Let S, T be Stonian spaces and $\pi : T \to S$ be a continuous surjection. A linear operator $u : C(T) \to C(S)$ is π-modular if $u(f \cdot g \circ \pi) = g \cdot u(f)$ for all $f \in C(T)$, $g \in C(S)$. We denote by $\mathscr{L}_\pi(C(T); C(S))$ the closed linear subspace of the Banach space $\mathscr{L}(C(T); C(S))$ of norm continuous linear operators, consisting of the π-modular operators.

We recall that a linear operator $A : E \to F$ (E, F Banach lattices) is positive if $Au \geqq 0$ for all $u \in E_+$. A positive linear operator $A : E \to F$ is order continuous if for every upward directed set Ω in E_+ with $\sup \Omega = u$, $A\Omega$ is upward directed and $\sup A\Omega = Au$. We denote by $\mathscr{L}^*(E; F)$ the space of norm bounded linear operators $E \to F$ that are expressible as the difference of two positive order continuous linear operators and refer to operators in $\mathscr{L}^*(E; F)$ simply as order continuous operators.

We shall need the following result:

2.2. Theorem (Haydon [2]). *A Banach lattice E is a 1-injective Banach lattice if and only if there exist Stonian spaces S, T and a continuous surjection $\pi : T \to S$ such that E is isometrically lattice isomorphic to $\mathscr{L}^*_\pi(C(T); C(S))$.*

3. A structure theorem

3.1. Definition. Let E be a 1-IBL such that $E = \mathscr{L}^*_\pi(C(T); C(S))$, with S, T, π as in Theorem 2.2 above. Given a linear function φ in E'_+ we define $\mu_\varphi : C(S) \to \mathbf{R}$ by

$$\mu_\varphi(f) = \sup\{\varphi(f \cdot u) : u \in \text{ball } E_+\}$$

for all $f \in C(S)_+$ where $(f \cdot u)(g) = f \cdot u(g)$ for all $g \in C(T)$, $f \in C(S)$, $u \in E$.

3.2. Proposition. *Let E be a 1-injective Banach lattice with $E = \mathscr{L}^*_\pi(C(T); C(S))$. For each $\varphi \in E'_+$ define μ_φ as above. Then*

(a) *μ_φ extends to a positive linear functional on $C(S)$, also denoted by μ_φ;*

(b) *$\varphi(u) \leqq \mu_\varphi(u 1_T)$ for all $u \in E_+$, where 1_T is the function $1_T(t) = 1$ for all $t \in T$;*

(c) *$\|\mu_\varphi\| = \mu_\varphi(1_s) = \|\varphi\|$.*

Proof. (a) Fix $\varepsilon > 0$ and choose $v \in \text{ball } E_+$ such that

$$\mu_\varphi(1_s) \leqq \varphi(v) + \varepsilon.$$

Then, by considering first of all $f \in \operatorname{ball} C(S)_+$ such that f takes only a finite number of values and extending by continuity to arbitrary $f \in \operatorname{ball} C(S)_+$ (see [2] section 3 and section 5), we have

$$\varphi(f \cdot u) \leqq \varphi(f \cdot v) + \varepsilon \|f\|$$

for all $u \in \operatorname{ball} E_+$, and all $f \in C(S)_+$.

Hence

$$\mu_\varphi(f \cdot u) \leqq \varphi(f \cdot v) + \varepsilon \|f\|$$

for all $f \in C(S)_+$.

Now given f, $g \in C(S)_+$, we have by definition

$$\mu_\varphi(f + g) \leqq \mu_\varphi(f) + \mu_\varphi(g)$$

and also

$$\begin{aligned}\mu_\varphi(f) + \mu_\varphi(g) &\leqq \varphi(f \cdot v) + \varphi(g \cdot v) + \varepsilon(\|f\| + \|g\|)\\ &= \varphi((f+g) \cdot v) + \varepsilon(\|f\| + \|g\|)\\ &\leqq \mu_\varphi(f+g) + \varepsilon(\|f\| + \|g\|)\\ &\leqq \mu_\varphi(f) + \mu_\varphi(g) + \varepsilon(\|f\| + \|g\|).\end{aligned}$$

As $\varepsilon > 0$ was arbitrary we have

$$\mu_\varphi(f + g) = \mu_\varphi(f) + \mu_\varphi(g).$$

Moreover for all $\alpha > 0$, $\alpha\mu_\varphi(f) = \mu_\varphi(\alpha f)$ is clear. Hence μ_φ extends to a positive linear functional on $C(S)$.

(b) Let $u \in E_+$ and $\Omega =$ support of $u 1_T$ in S. Set $h = (u 1_T) \cdot \chi_\Omega$. Then $h^{-1} \cdot u \in \operatorname{ball} E_+$, since if $-1_T \leqq g \leqq 1_T$, $-u 1_T \leqq u(g) \leqq u 1_T$. Hence

$$\begin{aligned}\mu_\varphi(u 1_T) &= \mu_\varphi(h)\\ &= \sup\{\varphi(h \cdot w) : w \in \operatorname{ball} E_+\}\\ &\geqq \varphi(h \cdot h^{-1} u) = \varphi(u).\end{aligned}$$

(c) This is clear from definitions. □

3.3. A Note. We recall the following criterion for weak compactness of subsets of $L_1(\nu)$-spaces, ν a finite measure: A subset A of $L_1(\nu)$ is relatively weakly compact if and only if for all $\sigma > 0$ there exists $M > 0$ such that $\|(|f| - M)^+\|_1 \leqq \sigma$ for all $f \in A$. Thus by the Kakutani representation theorem a bounded subset A of an AL-space F is relatively weakly compact if and only if for all $\sigma > 0$ there is $v \in F_+$ such that $\|(|u|-v)^+\| \leqq \sigma$ for all $u \in A$.

In particular let K be any compact space and denote by $M(K)$ the AL-space of all Radon measures on K, considered to be the strong dual of $C(K)$. For a bounded subset A of $M(K)$ it is shown in [6] that the following are equivalent:

(a) A is relatively weakly compact;

(b) For each weakly null sequence (f_n) in $C(K)$, $\lim_n \mu(f_n) = 0$ uniformly for μ in A;

(c) For each bounded sequence (f_n) in $C(K)$ satisfying $f_n \wedge f_m = 0$ $(m \neq n)$, $\lim_n \mu(f_n) = 0$ uniformly for μ in A;

(d) For each sequence (U_n) of disjoint open subsets of K, $\lim_n \mu(U_n) = 0$ uniformly for μ in A.

3.4. THEOREM. *Let E be a 1-injective Banach lattice and X any Banach space. Let $V: E \to X$ be a bounded linear operator. Then* either *V is an isomorphism on some copy of l_∞ in E* or *for all $\sigma > 0$ there exists an element $\psi \in E'_+$ such that $\|V_u\| \leqq \psi(|u|) + \sigma\|u\|$ for all $u \in E$.*

PROOF. We assume the Haydon representation $E = \mathcal{L}^*_\pi(C(T); C(S))$. For each $f \in X'$, $|V'f| \in E'_+$ and so with the notation of 3.1 above, let

$$A(V) = \{\mu_\varphi : \varphi = |V'f| \text{ and } f \in \text{ball } X'\}.$$

Then by Lemma 3.2, $A(V)$ is a norm bounded subset of $C(S)'_+$. Consider the following dichotomy:

(a) $A(V)$ is relatively weakly compact in $C(S)'$.

Then by the criterion in 3.3 above, given $\sigma > 0$ we can find $\lambda \in C(S)'_+$ such that

$$\|(\mu_\varphi - \lambda)^+\| \leqq \sigma \qquad \text{for all } \mu_\varphi \in A(V).$$

Hence for each $u \in E$,

$$\begin{aligned} \|Vu\| &= \sup\{f(Vu) : f \in \text{ball } X'\} \\ &\leqq \sup\{(|V'f|)(|u|) : f \in \text{ball } X'\} \\ &\leqq \sup\{\mu_\varphi(|u|1_T) : f \in \text{ball } X',\ \varphi = |V'f|\}. \end{aligned}$$

But by choice of λ and σ,

$$\sigma\|u\| \geqq (\mu_\varphi - \lambda)^+(|u|1_T) \geqq (\mu_\varphi - \lambda)(|u|1_T)$$

for all $\mu_\varphi \in A(V)$. Thus

$$\mu_\varphi(|u|1_T) \leqq \lambda(|u|1_T) + \sigma\|u\|, \qquad \mu_\varphi \in A(V);$$

and so

$$\| Vu \| \leqq \sup\{\mu_\varphi(|u| 1_T) : f \in \text{ball } X',\ \varphi = |V'f|\}$$

$$\leqq \lambda(|u| 1_T) + \sigma \| u \|, \qquad u \in E.$$

Define $\psi \in E'_+$ by $\psi(u) = \lambda(u 1_T)$, $u \in E$. Then

$$\| Vu \| \leqq \lambda(|u| 1_T) + \sigma \| u \| = \psi(|u|) + \sigma \| u \|, \qquad u \in E.$$

(b) $A(V)$ is *not* relatively weakly compact. Since $A(V)$ is a norm bounded subset of $C(S)'$, criterion (d) in 3.3 implies that we can find disjoint closed and open subsets (A_n) of the Stonian space S and $\varepsilon > 0$ such that

(1) $$\sup\{\mu_\varphi(e_n) : \mu_\varphi \in A(V)\} \geqq \tfrac{4}{3}\varepsilon > 0,$$

for all n where $e_n = \chi_{A_n}$, the characteristic function of A_n.

But for each $f \in \text{ball } X'$ and each n, if $\varphi = |V'f|$, we have

$$\mu_\varphi(e_n) = \sup\{|V'f|(e_n \cdot u) : u \in \text{ball } E_+\}$$

$$= \sup\{|V'f|(z) : 0 \leqq z \leqq e_n \cdot u,\ u \in \text{ball } E_+\}$$

so

(2) $$\mu_\varphi(e_n) = \sup\{|f(Vz)| : 0 \leqq z \leqq e_n \cdot u,\ u \in \text{ball } E_+\}.$$

Fix n. Then by (1) we can find $f_n \in \text{ball } X'$ such that if $\varphi_n = |V'f_n|$ then

(3) $$\mu_{\varphi_n}(e_n) > \varepsilon.$$

And hence by (2) and (3) we can find $u_n \in \text{ball } E_+$ such that

(4) $$\sup\{|f_n(Vz)| : 0 \leqq z \leqq e_n \cdot u_n\} > \tfrac{2}{3}\varepsilon.$$

Now by (4) we can find z_n, $0 \leqq z_n \leqq e_n \cdot u_n$ such that

(5) $$|f_n(Vz_n)| > \varepsilon/3.$$

Hence by (5) for all n,

(6) $$\| Vz_n \| = \sup\{f(Vz_n) : f \in \text{ball } X'\} > \varepsilon/3.$$

But now given any scalars (a_n) and $g \in C(T)$,

(7) $$\sum_n a_n z_n(g)(s) = a_m z_m(g)(s)$$

whenever $s \in A_m$, using the fact that $e_n \wedge e_m = 0\ (m \neq n)$ and $0 \leqq z_n \leqq e_n \cdot u_n$ for all n, so that $z_m \wedge z_n = 0\ (m \neq n)$. Now by (5)

$$\inf_n \|z_n\| \geqq \varepsilon/3 > 0.$$

Hence by (7)

$$\left\| \sum_n a_n z_n \right\| \geqq \sigma \sup_m |a_m|$$

so that the band generated by (z_n) in E is isomorphic to $l_\infty(\mathbf{N})$. By (5), V restricted to this band is not weakly compact. Hence by [3] prop. 2.f.4 there is a subspace of E isomorphic to $l_\infty(\mathbf{N})$ on which V is an isomorphism. □

3.5. Corollary. *Let X be a complemented subspace of an injective Banach lattice E. Then* either *X contains a subspace isomorphic to l_∞* or *X embeds as a subspace of an* AL-*space.*

Proof. Any injective Banach lattice embeds as a complemented subspace of a suitable 1-injective Banach lattice and so it suffices to consider complemented subspaces of 1-IBLs.

Let $P: E \to X$ be the projection. By 3.4 either P is an isomorphism on a copy of l_∞ in which case X contains l_∞ as an isomorphic subspace or given any $\sigma > 0$ we can find $\psi \in E'_+$ such that

$$\|Pu\| \leqq \psi(|u|) + \sigma\|u\|, \qquad u \in E.$$

In the latter case, if $u \in X = \text{range}(P)$ then

$$\|u\| = \|Pu\| \leqq \psi(|u|) + \sigma\|u\|;$$

so that $(1-\sigma)\|u\| \leqq \psi(|u|)$ for all $u \in X$. Take $\sigma = \frac{1}{3}$. Then $\|u\| \leqq \varphi(|u|)$ where $\varphi = \frac{3}{2}\psi \in E'_+$. Let

$$N = \{u \in E : \varphi(|u|) = 0\}.$$

Then N is a closed linear subspace of E that is also a lattice ideal in E. And so E/N is a normed lattice with norm $\|\hat{u}\|_1 = \varphi(|u|)$ for all $\hat{u} \in E/N$. Moreover $\|\cdot\|_1$ is additive on the canonical positive cone of E/N. Hence the completion E_φ of $(E/N, \|\cdot\|_1)$ is an AL-space. Finally, if $u \in X$, $\|u\| \leqq \varphi(|u|) \leqq \|\varphi\|\,\|u\|$ so that $\|u\| \leqq \|u\|_1 \leqq \|\varphi\|\,\|u\|$. Thus the quotient map $E \to E/N \hookrightarrow E_\varphi$ restricts to an isomorphic embedding of X into the AL-space E_φ. □

3.6. Definition. A Banach lattice E has order continuous norm (we say E is order continuous) if for every downward directed set $\{u_\alpha : \alpha \in A\}$ in E with $\inf\{u_\alpha : \alpha \in A\} = 0$, we have $\lim_\alpha \|u_\alpha\| = 0$.

3.7. THEOREM. *An order continuous injective Banach lattice is order isomorphic to an* AL-*space.*

PROOF. An order continuous Banach lattice F does not contain a subspace isomorphic to l_∞, [6]. Now F is, by hypothesis, a sublattice of an injective Banach lattice so that the lattice homomorphism constructed in the proof of 3.5 embeds F as a sublattice of E_φ. Hence F is a closed sublattice of an AL-space E_φ and hence F is order isomorphic to an AL-space. □

ACKNOWLEDGEMENTS

I would like to thank Dr. R. G. Haydon for many useful criticisms and much guidance regarding the contents of this paper. Financial support by the World University Service (UK) and Queen's College, Oxford made this investigation possible.

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